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Finite Energy Sum Rules*

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ABSTRACT

Finite energy sum rules for arbitrary scattering amplitudes are derived. They form consistency conditions that are imposed by analyticity alone. For amplitudes that decrease fast enough, they tend in the limit of infinite integration to the usual superconvergence relations. In our formulation, all Regge poles appear in the same form irrespective of their α-value. This helps in particular to resolve difficulties which appear in the superconvergence relations due to the uncertainty of the exact location of the leading singularity. The finite form of the sum rules makes them particularly useful in practical applications. They are a good tool for the determination of Regge parameters from low-energy data.

De Alfaro *et al.*[1] have derived superconvergence relations for amplitudes which decrease faster than ν^{-1}. We derive a new and more general set of sum rules that involve a finite integration of amplitudes. They apply to all amplitudes that obey dispersion relations and they form consistency conditions on the Regge expansion. Our new relations reduce to the usual superconvergence relations in the limit of infinite integration in the special case that the asymptotic behavior shows a sufficiently rapid decrease. The finite energy sum rules are very useful both for practical applications and for better understanding of the issues involved. For instance, in the usual formulation the exact position of the leading Regge trajectory plays a crucial role[2,3] in determining whether or not a superconvergence relation exists. However, in our finite energy sum rules this is not crucial at all since all Regge terms appear in the same form, whether or not they decrease faster than ν^{-1}.

For the sake of simplicity, we shall first assume that the high-energy behavior of scattering amplitudes is given by a sum of Regge pole terms. Cuts (which are continuous superpositions of poles) and the background integral will be treated later. Let us consider an anti-symmetric amplitude that obeys the unsubtracted dispersion relation

$$F(\nu) = \frac{2\nu}{\pi} \int_0^\infty \frac{\mathrm{Im}\, F(\nu')}{\nu'^2 - \nu^2}\, d\nu' \qquad (1)$$

If its leading Regge term has $\alpha < -1$, it will obey the superconvergence relation

$$\int_0^\infty \mathrm{Im}\, F(\nu)\, d\nu = 0 \qquad (2)$$

and therefore satisfy also the anti-subtracted dispersion relation

$$F(\nu) = \frac{2}{\nu\pi} \int_0^\infty \frac{\nu'^2 \, \mathrm{Im}\, F(\nu')}{\nu'^2 - \nu^2} \, d\nu' \quad . \qquad (3)$$

However, if the leading Regge term (but not the next one) is above -1, we can subtract it from F, and the resulting amplitude will obey a superconvergence relation. In order to simplify the calculations we use instead of the conventional $Q_{-\alpha-1}$ function of the Regge pole, the simple power ν^α that has the right high-energy behavior and obeys an unsubtracted dispersion relation:

$$R(\nu) = \frac{\beta(1 - e^{-i\pi\alpha})}{\sin \pi\alpha \, \Gamma(\alpha+1)} \, \nu^\alpha \quad , \qquad (4)$$

$$R(\nu) = \frac{2\nu}{\pi} \int_0^\infty \frac{\beta}{\Gamma(\alpha+1)} \frac{\nu'^\alpha}{\nu'^2 - \nu^2} \, d\nu' \qquad (-1 < \alpha < 1) \; . \qquad (5)$$

The amplitude F-R will satisfy the superconvergence relation:

$$\int_0^\infty \mathrm{Im}\,(F - R) \, d\nu = 0 \quad . \qquad (6)$$

Let us now derive the finite energy sum rules. We divide the poles into three classes: α_i stands for all poles which are above -1, α_j for all poles below -1, and α_k for any pole that happens to be at -1. These three classes enter into the superconvergence rule in three very different ways: The poles above -1 have to be subtracted from the integrand, the poles below -1 do not appear at all, and the residue of the pole at -1 appears on the right-hand side.

$$\int_0^\infty \left[\text{Im}\, F - \sum_{\alpha_i > -1} \frac{\beta_i}{\Gamma(\alpha_i+1)} \nu^{\alpha_i} \right] d\nu = c_k \quad . \qquad (7)$$

Each term on the left-hand side diverges if evaluated separately. We intend to write the relation in a manifestly convergent form that will also be suitable for practical applications. Therefore, we cut off the integration at some $\nu_{max} = N$ and express the high-energy behavior by Regge terms whose α is below -1:

$$\int_0^N \left[\text{Im}\, F - \sum_{\alpha_i > -1} \frac{\beta_i}{\Gamma(\alpha_i+1)} \nu^{\alpha_i} \right] d\nu + \int_N^\infty \sum_{\alpha_j < -1} \frac{\beta_j}{\Gamma(\alpha_j+1)} \nu^{\alpha_j} d\nu = \beta_k \ . \qquad (8)$$

The poles α_j below -1 have now entered the sum rule, but in a quite different manner from the poles α_i above -1. We also notice that all integrals are now convergent. Performing the integration, we find the following finite energy sum rule:

$$S_o(N) = \frac{1}{N} \int_0^N \text{Im}\, F\, d\nu = \sum_{\alpha_i > -1} \frac{\beta_i N^{\alpha_i}}{\Gamma(\alpha_i+2)} + \sum_{\alpha_j < -1} \frac{\beta_j N^{\alpha_j}}{\Gamma(\alpha_j+2)} + \beta_k N^{-1}$$

$$= \sum_{\text{all } \alpha} \frac{\beta N^{\alpha}}{\Gamma(\alpha+2)} \quad . \qquad (9)$$

There are two very important features of Eq. (9) that must be stressed:

(a) All Regge terms enter in the final equation in the same form, irrespective of whether α happens to be above, at, or below -1. This is a big advantage of our method because it eliminates the special role that the point -1 has in the usual treatment of the superconvergence relation.

This latter can, of course, be rederived from (9) by letting $N \to \infty$, if all α are below -1.

(b) The relative importance of successive terms in the finite energy sum rule (9) is the same as in the usual Regge expansion of the function F, i.e., if a secondary pole or a cut is unimportant in a high-energy fit above N, then this singularity is unimportant to exactly the same extent in the low-energy sum rule.

The meaning of our sum rule (9) is further elucidated when one realizes that it can also be derived in the following manner: One first uses the forward dispersion relation in order to compute the high-energy behavior of Re F. As an input, one uses the experimental data below N and the Regge fit to Im F above N. Afterwards, one does a Regge fit to Re F and checks the consistency of the two Regge fits. The consistency equations are identical to our finite energy sum rule (9). As a matter of fact, they give a family of such relations involving increasing moments of Im F,

$$S_n(N) = \frac{1}{N^{n+1}} \int_0^N \nu^n \, \mathrm{Im}\, F \, d\nu = \sum_{\text{all } \alpha_i} \frac{\beta_i \, N^{\alpha_i}}{(\alpha_i+n+1)\, \Gamma(\alpha_i+1)} \tag{10}$$

for all even n. One derives the sum rules for odd n by considering the (unmeasurable) J-parity amplitude. There has been some discussion about whether fixed poles exist at the integer J-values of the wrong signature. These poles would simply appear on the right-hand side of Eq. (10). Schwarz [4)] derived sum rules assuming that such fixed poles are absent, while Mandelstam and Wang [5)] recently showed that they might exist.

For clarity of exposition, we have assumed up to now that the amplitude can be written as a sum of Regge pole terms ν^{α}. Let us now drop this assumption and start with the general J-plane integral in the Khuri [6] representation

$$\mathrm{Im}\, F(\nu) = \int_{J_o - i\infty}^{J_o + i\infty} dJ\, a(J)\, \nu^J \tag{11}$$

where we choose J_o bigger than the leading singularity. Let us integrate ν^n Im F $(n = 0,1,2\ldots.)$ from zero to N and interchange the integrations over J and ν:

$$S_n(N) \equiv \frac{1}{N^{n+1}} \int_o^N \nu^n\, \mathrm{Im}\, F\, d\nu = \int_{J_o - i\infty}^{J_o + i\infty} \frac{dJ\, a(J)\, N^J}{J + n + 1} \quad . \tag{12}$$

We can now do the usual expansion into Regge poles, [7] Regge cuts, and a background integral further to the left, and we see once more that the whole family of finite energy integrals $S_n(N)$ is given by a Regge representation similar to that of F. We do not need any particular assumption about essential singularities or fixed poles in the J-plane in order to derive this family of finite energy sum rules, Eq. (12).

The finite energy sum rules, Eq. (10), are not only valid for antisymmetric functions, but also for symmetric ones and for the functions $f(\nu)$ which have a definite J-parity and have only the right-hand cut.

We can derive similar sum rules for negative n. For the amplitude $f(\nu)$ of definite J-parity, the formula analogous to Eq. (10) is

$$\int_0^N \frac{\mathrm{Im}\, f(\nu)}{\nu^{m+1}}\, d\nu - \sum_i \frac{\beta_i}{\Gamma(\alpha_i+1)} \frac{N^{\alpha_i - m}}{\alpha_i - m} = \begin{cases} \pi \dfrac{f^{(m)}(0)}{m!} & m \geq 0 \\ 0 & m < 0 \end{cases} \tag{13}$$

where $f^{(m)}$ is the m-th derivative of f.

The special case of Eq. (13) with $m = 0$ is particularly interesting. If the leading trajectory has $\alpha \geq 0$ then a symmetric amplitude will have to have a subtraction constant. This constant can be determined from Eq. (13) by using the Regge parameters. Alternatively, if this constant is known (for example, scattering length) it helps in determining the high-energy fit. The last argument was essentially used by Igi [8] in establishing the existence of the P' trajectory.

Our finite energy sum rule sheds light on the behavior of the usual superconvergence relation, Eq. (2), as the dominant Regge pole moves up through -1. We recall that the usual relation gives zero if the dominant pole is slightly below -1, it gives a finite number (namely, the residue) if the pole is exactly at -1, and it gives infinity if the pole is slightly above -1. Our finite relation for $\int_0^N \mathrm{Im}\, f\, d\nu$ always gives a term of the kind $c\, N^{\alpha+1}$ which is much smaller than c (but not zero) for α below $-1-(\log N)^{-1}$, about equal to c for α within $-1 \pm (\log N)^{-1}$, and much larger than c (but not infinite) for α above $-1+(\log N)^{-1}$. We see that the violent jumps from 0 to c to ∞ have disappeared due to the finite integration, and that the J-plane has become "coarse-grained" with an uncertainty of $\Delta J \sim (\log N)^{-1}$.

This discussion helps us to remove apparent problems that were recently raised. Muzinich [2] and Phillips [3] pointed out that the existence of double ρ cuts might spoil the $\Delta I = 2$ superconvergence relation of the B amplitude that was checked by many authors.[9] Since this cut will have $\alpha_c(0) = 2\alpha_\rho(0) - 1$ it seemed to be very crucial what the exact value of α_ρ is. We see from Eq. (8) that for the practical application it is not crucial at all whether $\alpha_c(0)$ would be slightly below or above zero. In any case, the deviations of the sum rule from zero should tell us the approximate strength of the cut or any other existent singularity.

The trouble with this particular example[9] (as with many other ones) is that the only way to check the sum rule is by summing over direct channel resonances. The results have such big ambiguities that it is very difficult to draw any convincing conclusion. However, there exists at least one clean case where good experimental data is available for the amplitude, namely the πN charge exchange at forward directions.[10] Let us now turn to this problem and illustrate with it the application of our technique.

Good data for $\text{Im}\, F_{CEX} = k(\sigma_{tot}(\pi^- p) - \sigma_{tot}(\pi^+ p))$ exist from threshold to 22 BeV. The amplitude converges quickly to its Regge fit, and above 3.7 BeV this fit falls within the error bars of the experimental points. We choose therefore $N = 3.7$ BeV in Eq. (9).

One has to realize that for the energy region above 3.7 BeV, for which direct-channel resonances still exist, the Regge expansion cannot be a convergent series (of non-integer powers). It must rather be an asymptotic series, because each term ν^α is regular at the resonance energy and cannot represent a resonance pole in ν. However, the asymptotic series does represent the smoothed-out contribution of all resonances, it only fails to represent the remaining wiggles. This remainder is given by the

Regge background integral. Above 3.7 BeV, these resonance wiggles are much smaller than the systematic experimental error, and below this point the extrema of the experimental curve converge to the Regge fit at a rate of $e^{-0.8(\nu/M)}$.

Summing the data[11] we obtain

$$\int_0^N \mathrm{Im}\, F\, d\nu = \int_0^N k\, \Delta\sigma\, d\nu - g^2 \frac{\mu^2 \pi}{2M^2} = 23 \pm 2 \text{ mb BeV}^2$$

where we have included the N-Born term. On the other hand, the parameters of the Regge pole as determined from the high-energy fit[12] lead to

$$\int_0^N \frac{\beta}{\Gamma(\alpha+1)} \nu^\alpha\, d\nu = \frac{\beta}{\Gamma(\alpha+2)} N^{\alpha+1} = 26 \pm 4 \text{ mb BeV}^2 \quad .$$

We see that the one-pole fit is consistent within error bars with the sum rule.

The error of $\pm$ 4 in the Regge contribution comes from the error of the Regge parameters as determined by the fit to the high-energy data alone. We see that the integral over the experimental values of $k\, \Delta\sigma$ gave an error that is smaller by 50%. Therefore, we should take it as an input in an overall one-pole fit. In that case, we find the best values of the parameters (using 1 BeV as the scale factor in ν^α):

$$\alpha_\rho = 0.70 \pm 0.05 \quad ,$$

$$\beta_\rho/\Gamma(\alpha+1) = 4.1 \pm 0.3 \text{ mb BeV} \quad .$$

The errors allow the possibility of the existence of additional singularities with the right quantum numbers. The situation is particularly simple if the additional singularity is a pole at $\alpha_\rho \approx -1$. The relative contribution of such a second ρ-pole can be as big as 15% at 3.7 BeV. On the other hand, if we choose $\alpha_{\rho'} \approx \alpha_\rho$, then we do not get any limits on the strength of the ρ'-pole, since one cannot distinguish between the two poles. In particular, we would not be able to distinguish in this analysis (at $t = 0$) between the ρ-pole and a ρ-P cut.

Let us emphasize that the finite energy sum rules should be taken into account in any high-energy analysis. A careful experimental analysis of scattering amplitudes should be made at low energies as well as at high energies to provide a better accuracy for the determination of the Regge behavior.

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